INTEGRAL METHODS FOR THE CALCULATION OF GAS FLOWS WITH STRONG SHOCK WAVES

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When a strong shock wave propagates through a gas the density of the gas increases significantly. The region of disturbed motion next to the wave may be considered as a peculiar boundary layer, which is in many ways analogous to the boundary layer in a viscous fluid*. In the calculations of gas motion in this layer behind the shock wave integral methods may be used, which are basically similar to those used in the theory of boundary layers in a viscous fluid. The integral relationships in various special cases have already been used by the author in the solutions of problems of flows with strong shock waves [1,2]. Below is given , in brief, a general approach to the use of integral methods in such problems together with new examples of solutions.

1. We shall consider gas flows with plane, cylindrical and spherical waves arising from the propagation of a shock wave into a stagnant gas. Let a gas be confined in some volume V between a shock wave and some surface located inside the region of motion and composed of the same gas particles (this surface will be referred to as the piston surface). To this gas we shall apply the laws of conservation of mass, momentum and energy. We shall denote by M, K and E the mass, momentum (more precisely the intègral of moduli of elementary momenta) and the energy of the gas, respectively, in the volume under consideration. Let us assume

$$M = \int\limits_V \rho dV, \qquad K = \int\limits_V \rho v dV, \qquad E = \int\limits_V \rho\left(\frac{v^2}{2} + e\right) dV \tag{1.1}$$

* G.G. Chernyi. *Boundary-layer method in problems of ideal and viscous gas motions with a surface discontinuity* Dissertation, Moscow State University, 1956.

Then from the laws of observation we obtain

$$\dot{M} = \rho^\circ \dot{R}^\circ S^\circ \quad \text{or} \quad M = \rho^\circ V^\circ + \text{const} \tag{1.2}$$

$$\dot{K} = p_* S_* - p^\circ S^\circ + \int_{S_*}^{S^\circ} p dS \tag{1.3}$$

$$\dot{E} = \rho^\circ \dot{R}^\circ S^\circ e^\circ + p_* \dot{R}_* S_* \tag{1.4}$$

where ρ is the density, v is the velocity, p is the pressure, e is the internal energy of a gas per unit mass, (for an ideal gas $e = p/(\gamma - 1)$ where γ is the ratio of specific heats), R and S denote the radius and the area of the surfaces bounding the chosen volume of gas, and V is the volume inside the surface S. Here the superscript $^\circ$ denotes the shock wave and the gas parameters in front of it, the subscript asterisk $_*$ denotes the second bounding surface and the gas parameters on it, and primes denote differentiation with respect to time t.

If we approximate the distribution of gas parameters along the radius or along the Lagrangian coordinate by some functions which contain these parameters, then the dependence of these parameters on time may be found by using Equations (1.2) to (1.4), and some additional conditions, in a manner similar to the integral method of boundary-layer theory. In particular, the differential equations of motion in their various approximate forms may serve as additional conditions. Naturally, if a sufficient number of other conditions is available it is not necessary to satisfy all the integral relationships (1.2) to (1.4); one or even two of them may not be satisfied.

We shall investigate some variants of the use of integral relationships.

2. First we shall make the simplest assumption, namely, that the pressure and the velocity are the same for all gas particles between the shock wave and the piston, i.e. they depend only on time. For definiteness we shall further assume that the gas is ideal with constant specific heats. The equations of conservation of momentum and of energy may be written in the form

$$M\dot{v} + \dot{M}v = (p - p^\circ) S^\circ \tag{2.1}$$

$$\frac{d}{dt}\left[\frac{Mv^2}{2} + \frac{p(V^\circ - V_*)}{\gamma - 1}\right] = \dot{M}e^\circ + p\dot{V}_* \tag{2.2}$$

where the quantity M is determined by Formula (1.2).

Two equations, (2.1) and (2.2), connect the three functions of time p,

v and V° for the known piston expansion $V_*(t)$. To determine these functions another relation between them is necessary. In [2] this additional relation was derived from the assumption that in this form of the integral method the gas velocity v is equal to the velocity of the gas immediately behind the shock wave, which in turn is determined by the relationships at the shock (or even by assuming that the velocity v be equal to the velocity of shock-wave propagation). On this assumption solutions were presented in [2] for the problem of a point explosion followed by piston expansion at constant velocity for the cases of plane and cylindrical waves, and for the approximately equivalent problems of the flow of a jet of large supersonic velocity past a thin blunt wedge and a thin blunt cone. Let us compare the exact solution for a piston expanding according to a power law and the solution obtained by using this method for $p^\circ = 0$ and $\gamma = 1.4$. Figure 1 shows in dashed lines the approximate values of ratios of the piston volume to the volume bounded by the shock wave and the ratios of the pressure on the piston to the pressure behind the shock wave p^*, corresponding to formulas

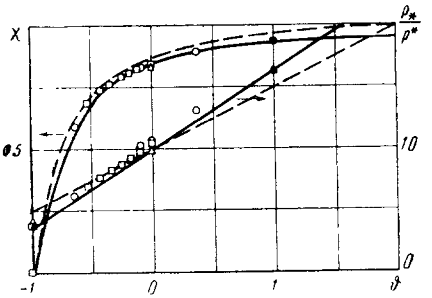

Fig. 1.

$$\frac{V_*}{V^\circ} = \frac{(1+\vartheta)\left(\frac{2\gamma}{\gamma+1} + \frac{\vartheta}{2}\right)}{\left(1+\frac{\vartheta}{2}\right)(\gamma+\vartheta)}, \quad \frac{p_*}{p^*} = 1 + \frac{\vartheta}{2} \qquad \left(\vartheta = \frac{2n}{\nu(n+1)}\right)$$

where $\nu = 1, 2, 3$ refer to flows with plane, cylindrical and spherical waves, respectively, n is the exponent in the expansion law of the piston $R_* \sim t^{n+1}$. Exact values were obtained from various sources, cited in [2].

Taking into account the simplicity of this approximate solution, its accuracy in the case under consideration may be considered to be satisfactory. A satisfactory accuracy is reached also in the solution by this method for the problem of the piston expanding with constant velocity ($n = 0$) for various values of the ratio of the sound velocity in a stagnant gas to the piston velocity.

3. Use of the "automodel" solutions [i.e. solutions based upon similarity relations]. Additional relations between the quantities which enter into the relationships (1.2) to (1.4) may be used, namely, those which exist between these quantities within the region of validity of similarity relations, which arise if the piston expands into a gas at rest with zero initial pressure according to the power law $R_* \sim t^{n+1}$.

We may, for example, assume that the volume distribution of the gas parameters in the region between the shock wave and the piston is determined by equations of the form

$$\rho = \rho^* \Omega \left(\frac{(1 - \lambda_m) V + \lambda_m V^\circ - V_*}{V^\circ - V_*} \right) \tag{3.1}$$

where the asterisk * denotes the parameters of the gas immediately behind the shock wave, $\Omega(\lambda)$ is the function taken from the similarity solution, where $\lambda = V/V^\circ$. This function, as well as the quantity λ_m (least value of λ), is a known function also on the exponent n in the law of piston expansion (and also on ν and γ). Using the assumed distributions we obtain

$$M = \rho^* (V^\circ - V_*) \mu(n), \qquad K = M v^* \varkappa(n), \qquad E = M v^{*2} \varepsilon(n)$$

$$p_* = \pi(n) p^*, \qquad \int_{S_*}^{S^\circ} p dS = p^* (S^\circ - S_*) \sigma(n)$$

where μ, κ, ϵ, π and σ are known functions of n. The substitution of these expressions in Equations (1.2) to (1.4) yields, for the known law of piston expansion $R_*(t)$, three relations between the quantities R°, n, ρ^*, v^* and p^* by which their dependence on time t is defined.

Two missing relationships may be taken from the three conditions at the shock wave, which connect ρ^*, v^*, p^* and R°. On the other hand we may use all three conditions at the shock wave. Then one of the three integral relationships (1.2) to (1.4) remains unsatisfied.

The method just described is analogous to the method of Kotchin and Loitsianski for the use of similarity solutions in the theory of boundary layers in a viscous fluid. However, it requires quite cumbersome computations, and is therefore replaced by a simpler method described in the next section.

4. Shock-layer method. As is known [1], the calculation of the gas motion behind strong shock waves may be carried through by means of representation of the solution in terms of Lagrangian variables in the form of power series in the parameter ϵ, which characterizes the ratio of densities of a gas in front and behind the wave. All the terms of these series are found from the equations by means of quadratures which contain the law of shock-wave propagation $R^\circ(t)$. For the determination of the function $R^\circ(t)$ the law of energy conservation (1.4) may be used. Here the function $R^\circ(t)$ must be represented also in the form of a series in ϵ. When substituting the series for R (R being an Eulerian coordinate), p and ρ in Equation (1.4) and, after a suitable transformation, equating the terms of the same power of ϵ on both sides of the equation, we obtain ordinary differential equations for the determination of the terms of the series for $R^{\circ}(t)$. This method may be made to assume the form of integral relationships, as was done in [2], if the approximate velocity and pressure distributions are used in the leading terms of the corresponding series in ϵ, if all the boundary conditions are satisfied, and if the function $R^{\circ}(t)$ which appears therein is determined from the integral energy relationship (1.4).

Following [2] we substitute the following approximate expressions of velocity and pressure:

$$v = \frac{2}{\gamma+1}\left(\dot{R}^{\circ} - \frac{a^{\circ 2}}{\dot{R}^{\circ}}\right) + O(\varepsilon)$$
$$p = p^{\circ} + \frac{2}{\gamma+1}\rho^{\circ}(\dot{R}^{\circ 2} - a^{\circ 2}) + \rho^{\circ}\frac{R^{\circ}\ddot{R}^{\circ}}{\nu} - \frac{\ddot{R}^{\circ}}{R^{\circ\nu-1}}m + O(\varepsilon) \qquad (4.1)$$

(a° is the sound velocity in the gas at rest, m is the Lagrangian coordinate proportional to the mass of a gas, contained inside the surface under consideration) in the integral energy relationship, disregarding in the formula for E the term

$$-\frac{\ddot{R}^{\circ}}{(\gamma-1)R^{\circ\nu-1}}\int_V m\,dV$$

which is of the order ϵ. As a result we obtain the following equations for the determination of the function R°:

$$\frac{d}{dt}\left\{\frac{1}{2}\left[\frac{2}{\gamma+1}\left(\dot{R} - \frac{a^{\circ 2}}{\dot{R}}\right)\right]^2 \rho^{\circ}R^{\nu} + \frac{p_*}{\gamma-1}(R^{\nu} - R_*^{\nu})\right\} = \frac{p^{\circ}}{\gamma-1}\frac{dR^{\nu}}{dt} + p_*\frac{dR_*^{\nu}}{dt} \qquad (4.2)$$

where

$$p_* = p^{\circ} + \frac{2}{\gamma+1}\rho^{\circ}(\dot{R}^2 - a^{\circ 2}) + \frac{\rho R\ddot{R}^{\circ}}{\nu}$$

For simplicity it is assumed that initially the gas occupies all space.

In [2] Equation (4.2) has been used to solve the problems of a piston moving at constant velocity (in this problem the solution coincides with the one obtained according to Section 2 of the present paper), of a piston moving according to a power law into a gas with zero initial pressure (see solid curves in Fig. 1), and of a strong explosion. In all these problems the approximate solutions come out in an elementary form and their coincidence with the exact solutions turns out to be quite satisfactory up to values ϵ = 0.2 – 0.3.

We shall present the solution of the problem of the explosion when the initial pressure is taken into account.

The basic equation (4.2) may be in this case integrated once, and thereupon it assumes the form

$$\frac{1}{2}\left[\frac{2}{\gamma+1}\left(\dot{R}+\frac{a^{\circ 2}}{\dot{R}}\right)\right]^2\rho^\circ R^\nu+\frac{p_*}{\gamma-1}(R^\nu-R_*^\nu)=\frac{E}{\omega}+\frac{p^\circ}{\gamma-1}R^\nu$$

where E is the energy of explosion (E is the energy, per unit area and per unit length of charge, respectively, for $\nu = 1$ and $\nu = 2$), $\omega = 2$, π, $4/3\,\pi$ for $\nu = 1, 2, 3$ respectively. When using the substitutions

$$\frac{a^{\circ 2}}{\dot{R}^2}=q, \qquad R^\nu=\frac{E}{\omega p^\circ}v$$

this equation and the expressions for p_* are reduced to the following:

$$\frac{v}{q^2}\frac{dq}{dv}=\frac{8\gamma}{(\gamma+1)^2}\left(\frac{1}{q}-1\right)\left(1-\frac{\gamma-1}{2\gamma}q\right)-\frac{2(\gamma-1)}{\gamma}\frac{1}{v} \tag{4.3}$$

$$\frac{p_*-p^\circ}{p^\circ}=\frac{\gamma-1}{v}-\frac{2\gamma(\gamma-1)}{(\gamma+1)^2}\frac{(1-q)^2}{q}$$

It is interesting to note that the system of relationships obtained does not contain ν, that is, it has the same form for the explosions of plane, linear and point charges. Consequently, the volume dependence of the propagation velocity of a shock wave (and, consequently, of all the gas parameters behind it) and of the pressure at the center of the explosion is the same for all three cases. The initial condition $q = 0$, $v = 0$ for the solution of Equation (4.3) corresponds to a singular point of this equation. In the neighborhood of this singularity the required solution has the following asymptotic form:

$$v=\frac{2(\gamma-1)(\gamma+1)^2}{\gamma(6\gamma-\gamma^2-1)}q, \qquad \frac{p_*-p^\circ}{p^\circ}=\frac{2\gamma^2-\gamma^3+3\gamma}{2(\gamma+1)^2}\frac{1}{q}$$

These first terms of asymptotic expansions approximately describe a strong explosion (without taking into account the initial gas pressure) and satisfactorily agree with the exact relations [2] up to the values $\epsilon = 0.2 - 0.3$. The dot-dash curve 3 in the Fig. 2 was obtained by numerical integration* of Equation (4.3), namely, of the function q from

$$l=\left(\frac{3}{4\pi}v\right)^{1/3}$$

* The calculations were carried through by G. Orlova and R. Burmistrova. The variable l was introduced for convenience of comparison with already existing exact solution for $\nu = 3$.

The dashed line 2 in this figure shows the results of the solution of the linearized problem of an explosion when the initial gas pressure is taken into account. The solid line 1 shows the values found for the complete numerical solution of the problem of a point explosion [3], line 4 shows the corresponding similarity solution. Unfortunately, data on the complete solution of the problem of the explosion of linear and plane charges does not exist to date. Figure 2 is evidence of the fact that the approximate solution of the problem of the explosion using the integral method is approximately of the same accuracy as the solution of the linearized problem, but differs from the latter by its simplicity.

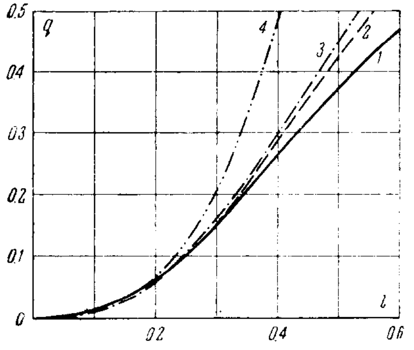

Fig. 2.

Also, the solutions presented for the problems of the piston and of the explosion lead to the conclusion that in these problems there is an approximate equivalence of motions of plane, cylindrical and spherical waves. When the law of plane cross-sections of supersonic aerodynamics is used it leads then to the equivalence of corresponding problems on streamlining of profiles and bodies of revolution (also slightly blunt bodies).

Next, we shall give a more complete example in which the integral method may be used.

We shall investigate the particular problem of a nonstationary supersonic source in a compressible gas, which arises in the study of shock tubes with an expanding nozzle.

An exact solution is known of the equations of steady motion of a compressible gas, in which the gas particles move along the rays originating at a singular point, and the values of all the gas parameters are the same on any concentric sphere about this point which is the source in the compressible gas. In the case of adiabatic motions of an ideal gas the solution may not be continued into the very center as it exists only outside the sphere of "critical" radius r_*, where the gas velocity v_* at any point of this sphere, which is called nucleus of the source, equals the sound velocity. If the pressure and the density (or the temperature) of the gas on the surface of a source nucleus are given and equal p_* and ρ_*, then there exist two continuous flows extending to infinity. In one of them the gas velocity decreases with the distance from the center to become zero at infinity; pressure and density on the other hand increase from

p_* and ρ_* to $p_T = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} p_*$ and $\rho_T = \left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}} \rho_*$ (case of a subsonic source)

In the other flow the gas velocity increases with the distance from the nucleus

from v to $\sqrt{\frac{\gamma+1}{\gamma-1}}\, v_*$ (case of a subsonic source)

at infinity, while pressure and density here correspondingly decrease to zero. This is the case of a supersonic source. A cylindrical source in a compressible gas may be treated in the same way. The two solutions here exist outside the nucleus, the shape of which is a circular cylinder.

If the pressure p° is greater than zero at infinity, but less that p_T, then a continuous flow is not possible and a shock wave arises in the stream. The flow here consists of a region of supersonic source flow adjoining the nucleus source, which by discontinuous change in the shock wave at a certain $r = r^*$ is converted into a region of subsonic source flow. The latter extends to infinity. With increase of pressure p° from zero the shock wave moves from infinity to the nucleus; when the shock wave approaches the nucleus its intensity decreases until at $p^\circ = p_T$ the shock wave becomes infinitely weak and coincides with the surface of the nucleus, while the flow everywhere becomes subsonic.

Let us consider now a tube consisting of cylindrical and conical portions.

Let the conical portion of the tube be filled with a homogeneous gas at rest, and let the parameters in the cylindrical section initially be such that subsequently the gas starts to flow from it at supersonic velocity into the conical section. The calculations of the motion generated under these conditions may approximately be reduced to a special case of the following problem of a nonsteady source.

Initially let the gas be at rest outside the sphere of radius r_0 and let its pressure be p° and its density ρ°. Inside the sphere between r_0 and $r_* < r_0$ initially the gas moves according to the law corresponding to the supersonic source. Its stagnation pressure and density equal p_T and ρ_T. Let us investigate the motion which arises from such an arbitrary discontinuity, under the condition that on the surface $r = r_*$ the velocity v_*, pressure p_* and density ρ_* remain constant. The investigation will be confined to the cases of those values of the characteristic parameters p°/p_T, ρ°/ρ_T (or T°/T_T), r_0/r_* (or M_0), γ and γ° for which the shock wave propagates in both directions from the contact discontinuity, and for which the asymptotically established discontinuity occurs at $r_{ju} > r_0$.

In formulating the equations which approximately describe the motion of the gas we shall apply the integral relationships.

As before let M, K and E denote respectively the mass, momentum and energy of a gas confined between the contact surface of the discontinuity and one of the shock waves. The equations of mass conservation in the region outside the contact surface (index "plus") and inside of it (index "minus") have the form

$$\frac{dM_+}{dt} = \rho^\circ \dot{R}^\circ S^\circ, \qquad \frac{dM_-}{dt} = \dot{\rho}_0 (v_0 - \dot{R}_0) S_0$$

or following integration

$$M_+ = \rho^\circ (V^\circ - V_{00}), \qquad M_- = qt - \int_{V_{00}}^{V_0} \rho_0 (V)\, dV \tag{4.4}$$

where q is the strength of the source, the indices "double zero" and "zero" denote, respectively, the values of the gas parameters in front of the shock waves, propagating to the outside and to the inside (following the particles). The quantities with index "double zero" are known constants, the quantities with index "zero" are known functions of R_0 (supersonic source), V_{00} is the volume inside the initial discontinuity.

The momentum equations have the form

$$\frac{dK_+}{dt} = p_* S_* - p^\circ S^\circ + \int_{S_*}^{S^\circ} p dS \qquad \frac{dK_-}{dt} = p_0 S_0 + \rho_0 v_0 (v_0 - \dot{R}_0) S_0 - p_* S_* + \int_{S_0}^{S_*} p dS \tag{4.5}$$

From the law of energy conservation we obtain

$$\frac{dE_+}{dt} = \rho^\circ \dot{R}^\circ S^\circ e^\circ + p_* \dot{R}_* S_*. \quad \frac{dE_-}{dt} = \rho_0 S_0 (v_0 - \dot{R}_0) \left(\frac{v_0^2}{2} + e_0\right) + p_0 v_0 S_0 - p_* \dot{R}_* S_* \tag{4.6}$$

We shall apply the law of integral relationships in its simplest form. We assume that $p_1 = p_2 = p_*(t)$, $v_1 = v_2 = \dot{R}_*(t)$. When using the integrals (4.4) Equations (4.5) to (4.6) thereupon assume the form

$$\rho^\circ (V^\circ - V_{00}) \ddot{R}_* + \rho^\circ S^\circ \dot{R}^\circ \dot{R}_* = (p_* - p^\circ) S^\circ$$

$$\left(qt - \int_{V_{00}}^{V_0} \rho_0 dV\right) \ddot{R}_* + \rho_0 S_0 (v_0 - \dot{R}_0)(\dot{R}_* - v_0) = (p_0 - p_*) S_0$$

$$\rho^\circ (V^\circ - V_{00}) R_* \ddot{R}_* + \rho^\circ S^\circ \dot{R}^\circ \frac{\dot{R}_*^2}{2} + \frac{V^\circ - V_*}{\gamma - 1} \dot{p}_* + \frac{S^\circ \dot{R}^\circ}{\gamma - 1} (p_* - p^\circ) - \frac{\gamma}{\gamma - 1} p_* S_* \dot{R}_* = 0$$

$$(qt - \int_{V_{**}}^{V_0} \rho dV)\dot{R}_*\ddot{R}_* + \frac{V_* - V_0}{\gamma - 1}\dot{p}_* = (q - p_0 S_0 \dot{R}_0)\frac{v_0^2 - \dot{R}_*^2}{2} + \frac{\gamma}{\gamma - 1}p_0 v_0 S_0 - $$
$$- \frac{\gamma}{\gamma - 1}p_* S_* \dot{R}_* + \frac{p_* - p_0}{\gamma - 1}S_0 \dot{R}_0 \qquad (4.7)$$

The four equations (4.7) contain an equal number of unknown functions R_0, R_*, R°, p_*. The initial conditions for the solution of system (4.7) have the following form:

$$R_0 = R_* = R^\circ = r_0, \quad p_* = p \quad \text{for } t = 0$$

where p is determined from the solution of the system of the following algebraic equations, which are obtained if in Equations (4.7) we assume $\dot{R}_* = \dot{p}_* = 0$, $S - 1$, and introduce notations $\dot{R}_0(0) = D_-$, $\dot{R}_*(0) = U$, $\dot{R}^\circ(0) = D_+$:

$$\rho^\circ D_+ U = p - p^\circ, \qquad \rho_0(v_0 - D_-)(U - v_0) = p_0 - p$$
$$\tfrac{1}{2}\rho^\circ D_+ U^2 + \frac{p - p^\circ}{\gamma - 1}D_+ - \frac{\gamma}{\gamma - 1}pU = 0$$
$$\tfrac{1}{2}\rho_0(v_0 - D_-)(v_0^2 - U^2) + \frac{\gamma}{\gamma - 1}p_0 v_0 - \frac{\gamma}{\gamma - 1}pU + \frac{p - p_0}{\gamma - 1}D_- = 0$$

The system of equations (4.7) may be integrated numerically for any set of determining parameters for which the accepted scheme of the flow takes place (i.e. for which $D_- > 0$).

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